

# 1+1+2 gravitational perturbations on LRS class II space-times: Decoupling GEM tensor harmonic amplitudes

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**Abstract.** This paper considers gauge-invariant and covariant gravitational perturbations on arbitrary vacuum *locally rotationally symmetric* (LRS) class II space-times. Ultimately, we derive four decoupled equations governing four specific combinations of the *gravito-electromagnetic* (GEM) 2-tensor harmonic amplitudes. We use the gauge-invariant and covariant 1+1+2 formalism which Clarkson and Barrett [1] developed for analysis of vacuum Schwarzschild perturbations. In particular we focus on the first-order 1+1+2 GEM system and use linear algebra techniques suitable for exploiting its structure. Consequently, we express the GEM system new 1+1+2 complex form by choosing new complex GEM tensors, which is conducive to decoupling. We then show how to derive a gauge-invariant and covariant decoupled equation governing a newly defined complex GEM 2-tensor. Finally, the GEM 2-tensor is expanded in terms of arbitrary tensor harmonics and linear algebra is used once again to decouple the system further into 4 real decoupled equations.

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## 1. Introduction

The gauge-invariant and covariant 1+1+2 formalism was first developed by Clarkson and Barrett [1] for an analysis of vacuum gravitational perturbations to a covariant Schwarzschild space-time. This was further developed in [2] who considered both scalar and electromagnetic (EM) perturbations to arbitrary *locally rotationally symmetric* (LRS) class II space-times [3, 4, 5], where they were able to derive generalized Regge-Wheeler [6] (RW) equations governing the 1+1+2 EM scalars,  $\mathcal{E}$  and  $\mathcal{B}$ . Subsequent to this, we also considered EM perturbations to LRS class II space-times [7, 8]. Therein, we used linear algebra techniques to show that the first-order 1+1+2 Maxwell's equations naturally decouple by choosing new complex variables. Consequently, we expressed Maxwell's equations in a new 1+1+2 complex form that is suited to decoupling. We reproduced the generalized RW result in a new complex form and further established that the EM 2-vectors,  $\mathcal{E}_\mu$  and  $\mathcal{B}_\mu$ , also decouple from the EM scalars. The EM 2-vectors were expanded into two polar perturbations  $\{\mathcal{E}_V, \bar{\mathcal{B}}_V\}$  and two axial perturbations  $\{\bar{\mathcal{E}}_V, \mathcal{B}_V\}$  using the arbitrary vector harmonic expansion developed in [1, 2]. Finally, we once again used linear algebra techniques, and derived four *real* decoupled equations governing the four combinations of the 2-vector harmonic amplitudes [8]. The precise combinations which decoupled were found to be, for the polar perturbations  $\{\mathcal{E}_V - \bar{\mathcal{B}}_V, \mathcal{E}_V + \bar{\mathcal{B}}_V\}$ , and for the axial perturbations  $\{\bar{\mathcal{E}}_V - \mathcal{B}_V, \bar{\mathcal{E}}_V + \mathcal{B}_V\}$ .

In this paper, we consider both gravitational and energy-momentum perturbations to arbitrary vacuum LRS class II space-times using the 1+1+2 formalism. The primary focus is with the first-order GEM system as it is well established to have remarkably similar mathematical structure to Maxwell's equations [9, 10]. We use similar techniques as in [8], which was successful to fully decouple the EM 2-vector harmonic components, and ultimately show that this is also successful for fully decoupling the GEM 2-tensor harmonic components.

In Section 2 we collate the important results arising from Clarkson and Barrett's 1+1+2 formalism and the background LRS class II space-time is reproduced from [2]. Also, the scalar and 2-vector harmonic expansion formalism is taken from [2], and we provide a new generalization of the *spherical* tensor harmonics developed in [1] to tensor harmonics. We use precisely the same notation as in [1, 2, 8] as well as introduce some new quantities which are well defined throughout. In Section 3, we carefully define the first-order perturbations (including the energy-momentum quantities) to be gauge-invariant according to the Sachs-Stewart-Walker Lemma [11, 12]. We proceed to write the first-order GEM system, conservation equations and the Ricci identities. In Section 5 we derive the decoupled equations and consider tensor harmonic expansions.

## 2. Preliminaries

The purpose of this section is to present the necessary results for the current series of papers on gravitational and energy-momentum perturbations to arbitrary vacuum LRS

class II space-times.

### 2.1. Clarkson and Barrett's 1+1+2 formalism

The 1+3 formalism is very well established (see for example [3, 9, 13]) whereby, a four-velocity  $u^\mu$  is defined such that it is both time-like and normalized ( $u^\alpha u_\alpha = -1$ ). Consequently, all quantities and governing equations are decomposed by projecting onto a 3-sheet which is orthogonal to  $u^\mu$ , and hence they are called 3-tensors, and in the time-like direction. The essential ingredient for Clarkson and Barrett's 1+1+2 formalism is to further decompose the 1+3 formalism by introducing a new “radial” vector  $n^\mu$  which is space-like and normalized ( $n^\alpha n_\alpha = 1$ ) and orthogonal to  $u^\mu$ . In this way, all 3-tensors may be further decomposed into 2-tensors which have been projected onto the 2-sheet orthogonal to both  $n^\mu$  and  $u^\mu$  and in the radial direction. The covariant derivative of the four-velocity in standard 1+3 notation is

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \frac{1}{3} \theta h_{\mu\nu} - u_\mu \dot{u}_\nu + \epsilon_{\mu\nu\alpha} \omega^\alpha, \quad (1)$$

where  $\nabla_\mu$  is the covariant derivative operator,  $\sigma_{\mu\nu}$  and  $\theta$  are the shear and expansion of the 3-sheets,  $h_{\mu\nu}$  is a tensor that projects onto the 3-sheets,  $\epsilon_{\mu\nu\sigma}$  is the Levi-Civita 3-tensor and  $\omega^\mu$  is the vorticity. Finally, the acceleration vector is  $\dot{u}^\mu$  where the “dot” derivative is defined  $\dot{X}_{\mu\dots\nu} := u^\alpha \nabla_\alpha X_{\mu\dots\nu}$  and  $X_{\mu\dots\nu}$  represents any quantity. Clarkson and Barrett irreducibly split these standard 1+3 quantities into 1+1+2 form according to

$$\dot{u}_\mu = \mathcal{A} n_\mu + \mathcal{A}_\mu, \quad (2)$$

$$\omega_\mu = \Omega n_\mu + \Omega_\mu, \quad (3)$$

$$\sigma_{\mu\nu} = \Sigma_{\mu\nu} - \frac{1}{2} N_{\mu\nu} \Sigma + 2 \Sigma_{(\mu} n_{\nu)} + \Sigma n_\mu n_\nu. \quad (4)$$

The 1+3 GEM fields are also decomposed,

$$E_{\mu\nu} = \mathcal{E}_{\mu\nu} - \frac{1}{2} N_{\mu\nu} \mathcal{E} + 2 \mathcal{E}_{(\mu} n_{\nu)} + \mathcal{E} n_\mu n_\nu, \quad (5)$$

$$H_{\mu\nu} = \mathcal{H}_{\mu\nu} - \frac{1}{2} N_{\mu\nu} \mathcal{H} + 2 \mathcal{H}_{(\mu} n_{\nu)} + \mathcal{H} n_\mu n_\nu, \quad (6)$$

where  $E_{\mu\nu}$  and  $H_{\mu\nu}$  are respectively the electric and magnetic parts of the Weyl tensor,  $C_{\mu\nu\sigma\tau}$ . In a similar fashion, the 3-covariant derivative ( $D_\mu$ ) of the radial vector is decomposed into 1+1+2 form according to

$$D_\mu n_\nu = n_\mu a_\nu + \frac{1}{2} \phi N_{\mu\nu} + \xi \epsilon_{\mu\nu} + \zeta_{\mu\nu}, \quad (7)$$

where  $\zeta_{\mu\nu}$  and  $\phi$  are respectively the shear and expansion of the 2-sheets,  $N_{\mu\nu}$  is a tensor that projects onto the 2-sheets,  $\xi$  represents the twisting of the sheet and  $\epsilon_{\mu\nu}$  is the Levi-Civita 2-tensor. Also, the acceleration 2-vector is  $a_\mu := \hat{n}_\mu$  where the “hat” derivative is defined  $\hat{W}_{\mu\dots\nu} := n^\alpha D_\alpha W_{\mu\dots\nu}$  and  $W_{\mu\dots\nu}$  represents a 3-tensor. Finally, the “dot” derivative of the radial normal is also split according to

$$\dot{n}_\mu = \mathcal{A} u_\mu + \alpha_\mu. \quad (8)$$

Therefore, the irreducible set of 1+1+2 quantities, and in accord with standard terminology, is

$$\begin{aligned} \text{scalars: } & \{\mathcal{A}, \phi, \Sigma, \theta, \mathcal{E}, \mathcal{H}, \Lambda, \xi, \Omega\}, \\ \text{2-vectors: } & \{a^\mu, \alpha^\mu, \Omega^\mu, \mathcal{A}^\mu, \Sigma^\mu, \mathcal{E}^\mu, \mathcal{H}^\mu\}, \\ \text{2-tensors: } & \{\Sigma_{\mu\nu}, \zeta_{\mu\nu}, \mathcal{E}_{\mu\nu}, \mathcal{H}_{\mu\nu}\}, \end{aligned} \quad (9)$$

where the cosmological constant ( $\Lambda$ ) has also been included. Furthermore, the energy-momentum quantities, heat-flux and anisotropic pressure, become respectively [2],

$$q^\mu = \mathcal{Q} n^\mu + \mathcal{Q}^\mu, \quad (10)$$

$$\pi_{\mu\nu} = \Pi_{\mu\nu} - \frac{1}{2} N_{\mu\nu} \Pi + 2 \Pi_{(\mu} n_{\nu)} + \Pi n_\mu n_\nu. \quad (11)$$

Thus, the irreducible 1+1+2 energy-momentum quantities are

$$\text{scalars: } \{\mu, p, \mathcal{Q}, \Pi\}, \quad \text{2-vectors: } \{\mathcal{Q}^\mu, \Pi^\mu\} \quad \text{and} \quad \text{2-tensor: } \{\Pi_{\mu\nu}\}. \quad (12)$$

where  $\mu$  is the mass-energy density and  $p$  is the isotropic pressure.

## 2.2. Background Vacuum LRS class II space-time

The background comprises the most general vacuum LRS class II space-time and is defined by six non-vanishing LRS class II scalars

$$\text{LRS class II : } \{\mathcal{A}, \phi, \Sigma, \theta, \mathcal{E}, \Lambda\}. \quad (13)$$

The background Ricci identities for both  $u^\mu$  and  $n^\mu$  and the Bianchi identities yields a set of evolution and propagation equations governing these scalars. They were first presented in [1] for a covariant Schwarzschild space-time and generalized to non-vacuum LRS class II space-times in [2] for which we reproduce them here for the vacuum case,

$$\left(\mathcal{L}_n + \frac{1}{2}\phi\right)\phi + \left(\Sigma - \frac{2}{3}\theta\right)\left(\Sigma + \frac{1}{3}\theta\right) + \mathcal{E} + \frac{2}{3}\Lambda = 0, \quad (14)$$

$$\left(\mathcal{L}_n + \frac{3}{2}\phi\right)\Sigma - \frac{2}{3}\mathcal{L}_n\theta = 0, \quad (15)$$

$$\left(\mathcal{L}_n + \frac{3}{2}\phi\right)\mathcal{E} = 0, \quad (16)$$

$$\left(\mathcal{L}_u - \frac{1}{2}\Sigma + \frac{1}{3}\theta\right)\phi + \mathcal{A}\left(\Sigma - \frac{2}{3}\theta\right) = 0, \quad (17)$$

$$\left(\mathcal{L}_u - \frac{1}{2}\Sigma + \frac{1}{3}\theta\right)\left(\Sigma - \frac{2}{3}\theta\right) + \mathcal{A}\phi + \mathcal{E} + \frac{2}{3}\Lambda = 0, \quad (18)$$

$$\left(\mathcal{L}_u - \frac{3}{2}\Sigma + \theta\right)\mathcal{E} = 0, \quad (19)$$

$$\left(\mathcal{L}_n + \mathcal{A} - \frac{1}{2}\phi\right)\mathcal{A} - \frac{3}{2}\left(\mathcal{L}_u + \frac{1}{2}\Sigma + \frac{2}{3}\theta\right)\Sigma - \frac{3}{2}\mathcal{E} + \Lambda = 0, \quad (20)$$

$$\left(\mathcal{L}_u + \Sigma + \frac{1}{3}\theta\right)\left(\Sigma + \frac{1}{3}\theta\right) - (\mathcal{L}_n + \mathcal{A})\mathcal{A} + \mathcal{E}, \quad (21)$$

$$\delta_\mu \mathcal{E} = \delta_\mu \phi = \delta_\mu \mathcal{A} = \delta_\mu \theta = \delta_\mu \Sigma = 0, \quad (22)$$

where  $\delta_\mu$  is the covariant 2-derivative associated with the 2-sheet. Moreover, in addition to the “dot” and “hat” derivatives, we will also use the Lie derivative,  $\mathcal{L}_u$  and  $\mathcal{L}_n$ ,

(where, for example, the standard definition can be found in [14]). This allows us to neatly express the equations using covariant differential operators. Since the system (14)-(22) considers only scalars, they simply become usual directional derivatives in this case and are equivalent to the “dot” and “hat” derivatives,

$$\mathcal{L}_u \psi = \dot{\psi} \quad \text{and} \quad \mathcal{L}_n \psi = \hat{\psi}. \quad (23)$$

Furthermore, for a 2-vector  $\psi_\mu$  and 2-tensor  $\psi_{\mu\nu}$ , they are related as follows,

$$\left(\mathcal{L}_n - \frac{1}{2}\phi\right)\psi_{\bar{\mu}} = \hat{\psi}_{\bar{\mu}} \quad \text{and} \quad (\mathcal{L}_n - \phi)\psi_{\bar{\mu}\bar{\nu}} = \hat{\psi}_{\bar{\mu}\bar{\nu}}, \quad (24)$$

$$\left(\mathcal{L}_u + \frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\psi_{\bar{\mu}} = \dot{\psi}_{\bar{\mu}} \quad \text{and} \quad \left(\mathcal{L}_u + \Sigma - \frac{2}{3}\theta\right)\psi_{\bar{\mu}\bar{\nu}} = \dot{\psi}_{\bar{\mu}\bar{\nu}}. \quad (25)$$

It is also convenient to introduce five more definitions for the 2-gradients of the LRS class II scalars that arise in (22). Three of these arise in [1],

$$X_\mu := \delta_\mu \mathcal{E}, \quad Y_\mu := \delta_\mu \phi \quad \text{and} \quad Z_\mu := \delta_\mu \mathcal{A}, \quad (26)$$

and two new definitions are made to account for the additional complications of an *arbitrary* LRS class II background,

$$V_\mu := \delta_\mu \left(\Sigma + \frac{1}{3}\theta\right) \quad \text{and} \quad W_\mu := \delta_\mu \left(\Sigma - \frac{2}{3}\theta\right). \quad (27)$$

Finally, as in [2] we also find it useful to work with the extrinsic curvature and it also comes with evolution and propagation equations,

$$K = \frac{1}{4}\phi^2 - \frac{1}{4}\left(\Sigma - \frac{2}{3}\theta\right)^2 - \mathcal{E} + \frac{1}{3}\Lambda, \quad (28)$$

$$(\mathcal{L}_n + \phi)K = 0 \quad \text{and} \quad \left(\mathcal{L}_u - \Sigma + \frac{2}{3}\theta\right)K = 0. \quad (29)$$

### 2.3. Harmonic Expansions

The *spherical harmonic expansions* for 1+1+2 scalars, 2-vectors and 2-tensors were first presented in [1] for the specific Schwarzschild case. This was subsequently generalized to *harmonic expansions* for both scalars and 2-vectors in [2]. In this section, we reproduce the necessary results from [2] as well as include a new generalization of the 2-tensor *spherical* harmonics in [1] to 2-tensor harmonics. Dimensionless sheet harmonic functions  $Q$  (defined on the background) are defined

$$\delta^2 Q := -\frac{k^2}{r^2} Q \quad \text{and} \quad \hat{Q} = \dot{Q} = 0, \quad (30)$$

where  $k^2$  is real and the 2-Laplacian is defined  $\delta^2 := \delta^\alpha \delta_\alpha$ . The scalar function  $r$  is defined by the following covariant equations

$$\left(\mathcal{L}_n - \frac{1}{2}\phi\right)r = 0, \quad \left(\mathcal{L}_u + \frac{1}{2}\Sigma - \frac{1}{3}\theta\right)r = 0 \quad \text{and} \quad \delta_\mu r = 0. \quad (31)$$

Now any first-order scalar function can be expanded as

$$\psi = \sum_k \psi_S^{(k)} Q^{(k)} = \psi_S Q, \quad (32)$$

where  $\psi_s$  is the scalar harmonic amplitude and the summation over  $k$  is implicit in the last equality. Similarly, all vectors are expanded in terms of even ( $Q_\mu$ ) and odd ( $\bar{Q}_\mu$ ) parity vector harmonics which are defined respectively

$$Q_\mu = r \delta_\mu Q \quad \rightarrow \quad \delta^2 Q_\mu = \left( K - \frac{k^2}{r^2} \right) Q_\mu, \quad (33)$$

$$\bar{Q}_\mu = r \epsilon_\mu^\alpha \delta_\alpha Q \quad \rightarrow \quad \delta^2 \bar{Q}_\mu = \left( K - \frac{k^2}{r^2} \right) \bar{Q}_\mu. \quad (34)$$

The vector harmonics are orthogonal ( $Q^\alpha \bar{Q}_\alpha = 0$ ) and they have the following properties:  $\bar{Q}_\mu = \epsilon_\mu^\alpha Q_\alpha$  and  $Q_\mu = -\epsilon_\mu^\alpha \bar{Q}_\alpha$ . Thus any first-order vector may be expanded according to

$$\psi_\mu = \sum_k \psi_V^{(k)} Q_\mu^{(k)} + \bar{\psi}_V^{(k)} \bar{Q}_\mu^{(k)} = \psi_V Q_\mu + \bar{\psi}_V \bar{Q}_\mu, \quad (35)$$

where similarly  $\psi_V$  and  $\bar{\psi}_V$  are the vector harmonic amplitudes and the summation in the last quantity is implicit. Also note that the 2-Laplacian acting on the vector harmonics in (33)-(34) is written in terms of the Gaussian curvature here, whereas in [2] they use a further constraint of  $K = 1/r^2$  which amounts to choosing a particular normalization that was convenient for their analysis.

We now present a generalization of the spherical tensor harmonics presented in [1] to tensor harmonics in arbitrary LRS class II space-times. The even and odd tensor harmonics are defined respectively

$$Q_{\mu\nu} = r^2 \delta_{\{\mu} \delta_{\nu\}} Q, \quad \delta^2 Q_{\mu\nu} = \left( 4K - \frac{k^2}{r^2} \right) Q_{\mu\nu}, \quad (36)$$

$$\bar{Q}_{\mu\nu} = r^2 \epsilon_{\alpha\{\mu} \delta^\alpha \delta_{\nu\}} Q, \quad \delta^2 \bar{Q}_{\mu\nu} = \left( 4K - \frac{k^2}{r^2} \right) \bar{Q}_{\mu\nu}, \quad (37)$$

where the “curly” brackets indicate the part that is symmetric and trace-free with respect to the 2-sheet. These are orthogonal ( $Q^{\alpha\beta} \bar{Q}_{\alpha\beta} = 0$ ) and have the following properties:  $Q_{\mu\nu} = \epsilon_{(\mu}^\alpha \bar{Q}_{\nu)\alpha}$  and  $\bar{Q}_{\mu\nu} = -\epsilon_{(\mu}^\alpha Q_{\nu)\alpha}$ . Therefore, all first-order tensors may now be expanded in terms of tensor harmonics according to

$$\psi_{\mu\nu} = \sum_k \psi_T^{(k)} Q_{\mu\nu}^{(k)} + \bar{\psi}_T^{(k)} \bar{Q}_{\mu\nu}^{(k)} = \psi_T Q_{\mu\nu} + \bar{\psi}_T \bar{Q}_{\mu\nu}, \quad (38)$$

where in accord with usual terminology,  $\psi_T$  and  $\bar{\psi}_T$  are the tensor harmonic amplitudes and again the summation in the last equality is implicit. We also have the following relationships which also generalize those presented in [1],

$$\delta^\alpha \psi_{\mu\alpha} = \frac{r}{2} \left( 2K - \frac{k^2}{r^2} \right) \left( \psi_T Q_\mu - \bar{\psi}_T \bar{Q}_\mu \right), \quad (39)$$

$$\epsilon_{\{\mu}^\alpha \delta^\beta \psi_{\beta\}\alpha} = \frac{r}{2} \left( 2K - \frac{k^2}{r^2} \right) \left( \bar{\psi}_T Q_\mu + \psi_T \bar{Q}_\mu \right). \quad (40)$$

### 3. The Gravitational and Energy-Momentum Perturbations

We now consider both gravitational and energy-momentum perturbations to the background LRS class II space-time defined in Section 2.2. In agreement with traditional

practice we let all gravitational and energy-momentum quantities that vanish on the background LRS class II space-time simply become quantities of first-order ( $\epsilon$ ), i.e.

$$\text{first-order scalars: } \{\mathcal{H}, \xi, \Omega, \mu, p, \mathcal{Q}, \Pi\} = \mathcal{O}(\epsilon), \quad (41)$$

$$\text{first-order 2-vectors: } \{a^\mu, \alpha^\mu, \Omega^\mu, \mathcal{A}^\mu, \Sigma^\mu, \mathcal{E}^\mu, \mathcal{H}^\mu, \mathcal{Q}^\mu, \Pi^\mu\} = \mathcal{O}(\epsilon), \quad (42)$$

$$\text{first-order 2-tensors: } \{\Sigma_{\mu\nu}, \zeta_{\mu\nu}, \mathcal{E}_{\mu\nu}, \mathcal{H}_{\mu\nu}, \Pi_{\mu\nu}\} = \mathcal{O}(\epsilon). \quad (43)$$

The first-order quantities given in (41)-(43) are all gauge-invariant under infinitesimal coordinate transformations, or more formally due to the Sachs-Stewart-Walker Lemma [11, 12], as their corresponding background terms vanish. Furthermore, there is also the issue of choosing a particular frame in the perturbed space-time (i.e. choosing the first-order four-velocity and radial vector) as also discussed in [1]. In general, the first-order gauge-invariant 1+1+2 quantities will not be frame invariant as they naturally depend on this choice since their underlying definitions are typically just projections and contractions with the four-velocity and radial vector.

Now consider some perturbed quantity,  $\tilde{\psi}$ , this is expanded to first-order according to

$$\tilde{\psi} = \psi + \delta\psi, \quad (44)$$

where  $\psi$  is the corresponding background value and  $\delta\psi$  is the corresponding first-order part (and  $\delta$  is not to be confused with the covariant 2-derivative  $\delta_\mu$ ). Therefore, there are five LRS class II scalars which do not vanish on the background, and thus, they will experience first-order increments given by

$$\{\delta\mathcal{A}, \delta\phi, \delta\Sigma, \delta\theta, \delta\mathcal{E}\} = \mathcal{O}(\epsilon). \quad (45)$$

Furthermore, these five first-order scalars (45) are not gauge-invariant under the Sachs-Stewart-Walker Lemma. However, as initiated in [1], the 2-gradient of these scalars does vanish on the background according to (22), and therefore, they become gauge-invariant quantities of first-order,

$$\text{first-order 2-vectors: } \{V_\mu, W_\mu, X_\mu, Y_\mu, Z_\mu\} = \mathcal{O}(\epsilon). \quad (46)$$

Throughout the remainder of this paper, every equation is written in a purely gauge-invariant way. This is predominately achieved by writing everything explicitly in terms of the quantities defined in (41)-(42) and (46), otherwise, it is ensured that particular combinations of gauge-variant quantities are written as one combined gauge-invariant quantity.

#### 4. The first-order Bianchi and Ricci Identities

The equations governing the first-order gauge-invariant 1+1+2 variables are found by decomposing the Ricci identities for both  $u^\mu$  and  $n^\mu$ , the once contracted Bianchi identities (GEM system) and the twice contracted Bianchi identities.

#### 4.1. Twice-contracted Bianchi Identities

In this paper we consider the first-order energy-momentum quantities as a known source that is capable of physically perturbing the background space-time giving rise to first-order gravitational fields. Therefore, we begin with the conservation of mass equations as they will indicate how these first-order energy-momentum quantities propagate and evolve,<sup>‡</sup>

$$(\mathcal{L}_u + \theta) \mu + (\mathcal{L}_n + 2\mathcal{A} + \phi) \mathcal{Q} + \delta^\alpha \mathcal{Q}_\alpha + p\theta + \frac{3}{2} \Pi \Sigma = 0, \quad (47)$$

$$\left( \mathcal{L}_u + \Sigma + \frac{4}{3} \theta \right) \mathcal{Q} + (\mathcal{L}_n + \mathcal{A})p + \mu \mathcal{A} + \delta^\alpha \Pi_\alpha + \left( \mathcal{L}_n + \mathcal{A} + \frac{3}{2} \phi \right) \Pi = 0, \quad (48)$$

$$(\mathcal{L}_u + \theta) \mathcal{Q}_{\bar{\mu}} + (\mathcal{L}_n + \mathcal{A} + \phi) \Pi_{\bar{\mu}} + \delta_{\bar{\mu}} \left( p - \frac{1}{2} \Pi \right) + \delta^\alpha \Pi_{\mu\alpha} = 0. \quad (49)$$

.

#### 4.2. Gravito-electromagnetism

The 1+1+2 GEM system is of prime importance as this paper is predominately focused on decoupling the GEM 2-tensor harmonic amplitudes. The once contracted Bianchi identities may be written in terms of the Weyl and energy-momentum tensor according to

$$B_{\nu\sigma\tau} := \nabla^\mu C_{\mu\nu\sigma\tau} - [\nabla_{[\sigma} T_{\tau]\nu} + \frac{1}{3} g_{\nu[\sigma} \nabla_{\tau]} T] = 0. \quad (50)$$

Before proceeding with the linearized system, we momentarily discuss the fully non-linear 1+3 GEM system, for which it is important to note that it is invariant under the simultaneous transformation  $E_{\mu\nu} \rightarrow H_{\mu\nu}$  and  $H_{\mu\nu} \rightarrow -E_{\mu\nu}$  (in the absence of sources). In a recent paper [7], we used linear algebra techniques to show that the most natural way to decouple a system with these particular invariance properties is to choose new complex dynamical variables. This has also been discussed elsewhere; for example, see [10] where they introduce a complex tensor defined  $\mathcal{I}_{\mu\nu} := E_{\mu\nu} \pm i H_{\mu\nu}$  (where  $i$  is the complex number). It was also this reason why we successfully decoupled the EM 2-vector harmonic amplitudes in [8].

We now turn the attention to the first-order 1+1+2 GEM system which reduces to§

$$\delta \left[ \left( \mathcal{L}_n + \frac{3}{2} \phi \right) \mathcal{E} \right] + \delta^\alpha \mathcal{E}_\alpha = \Re[\mathcal{G}], \quad (51)$$

$$\left( \mathcal{L}_n + \frac{3}{2} \phi \right) \mathcal{H} + \delta^\alpha \mathcal{H}_\alpha + 3 \mathcal{E} \Omega = \Im[\mathcal{G}], \quad (52)$$

$$\delta \left[ \left( \mathcal{L}_u - \frac{3}{2} \Sigma + \theta \right) \mathcal{E} \right] - \epsilon^{\alpha\beta} \delta_\alpha \mathcal{H}_\beta = \Re[\mathcal{F}], \quad (53)$$

<sup>‡</sup> These are derived as follows, (47) from  $u^\alpha \nabla^\beta T_{\alpha\beta} = 0$ ; (48) from  $n^\alpha \nabla^\beta T_{\alpha\beta} = 0$ ; (49) from  $\nabla^\alpha T_{\bar{\mu}\alpha} = 0$ .

<sup>§</sup> These are derived as follows, (51) from  $u^\alpha u^\beta n^\gamma B_{\alpha\beta\gamma} = 0$ ; (52) from  $\epsilon^{\beta\gamma} u^\alpha B_{\alpha\beta\gamma} = 0$ ; (53) from  $u^\alpha n^\beta n^\gamma B_{\beta\gamma\alpha} = 0$ ; (54) from  $\epsilon^{\alpha\beta} n^\gamma B_{\gamma\alpha\beta} = 0$ ; (55) from  $u^\beta u^\gamma B_{\bar{\mu}\beta\gamma} = 0$ ; (56) from  $\epsilon_{\bar{\mu}}^{\beta\gamma} u^\alpha B_{\alpha\beta\gamma} = 0$ ; (57) from  $n^\nu u^\gamma B_{(\bar{\mu}\nu)\gamma} = 0$ ; (58) from  $n^\nu \epsilon_{(\bar{\mu}}^{\alpha\beta} B_{\nu)\alpha\beta} = 0$ ; (59) from  $u^\alpha B_{(\bar{\mu}\bar{\nu})\alpha} = 0$ ; (60) from  $\epsilon_{(\bar{\mu}}^{\alpha\beta} B_{\bar{\nu})\alpha\beta} = 0$ .

$$\left(\mathcal{L}_u - \frac{3}{2}\Sigma + \theta\right)\mathcal{H} + \epsilon^{\alpha\beta}\delta_\alpha\mathcal{E}_\beta + 3\mathcal{E}\xi = \Im[\mathcal{F}], \quad (54)$$

$$(\mathcal{L}_n + \phi)\mathcal{E}_{\bar{\mu}} + \delta^\alpha\mathcal{E}_{\mu\alpha} - \frac{1}{2}X_\mu + \frac{3}{2}\Sigma\epsilon_\mu^\alpha\mathcal{H}_\alpha + \frac{3}{2}\mathcal{E}a_\mu = \Re[\mathcal{G}_\mu], \quad (55)$$

$$(\mathcal{L}_n + \phi)\mathcal{H}_{\bar{\mu}} + \delta^\alpha\mathcal{H}_{\mu\alpha} - \frac{1}{2}\delta_\mu\mathcal{H} - \frac{3}{2}\Sigma\epsilon_\mu^\alpha\mathcal{E}_\alpha + \frac{3}{2}\mathcal{E}\epsilon_\mu^\alpha(\Sigma_\alpha + \epsilon_\alpha^\beta\Omega_\beta) = \Im[\mathcal{G}_\mu], \quad (56)$$

$$\left(\mathcal{L}_u - \Sigma + \frac{2}{3}\theta\right)\mathcal{E}_{\bar{\mu}} - \epsilon_\mu^\alpha\delta^\beta\mathcal{H}_{\alpha\beta} - \frac{1}{2}\epsilon_\mu^\alpha[\delta_\alpha\mathcal{H} - (2\mathcal{A} - \phi)\mathcal{H}_\alpha] + \frac{3}{2}\mathcal{E}\alpha_\mu = \Re[\mathcal{F}_\mu], \quad (57)$$

$$\left(\mathcal{L}_u - \Sigma + \frac{2}{3}\theta\right)\mathcal{H}_{\bar{\mu}} + \epsilon_\mu^\alpha\delta^\beta\mathcal{E}_{\alpha\beta} + \frac{1}{2}\epsilon_\mu^\alpha[X_\alpha - (2\mathcal{A} - \phi)\mathcal{E}_\alpha] + \frac{3}{2}\mathcal{E}\epsilon_\mu^\alpha\mathcal{A}_\alpha = \Im[\mathcal{F}_\mu], \quad (58)$$

$$\left(\mathcal{L}_u + \frac{5}{2}\Sigma + \frac{1}{3}\theta\right)\mathcal{E}_{\bar{\mu}\bar{\nu}} + \epsilon_{(\mu}^\alpha\left(\mathcal{L}_n + 2\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{H}_{\nu)\alpha} - \epsilon_{\{\mu}^\alpha\delta_{|\alpha|}\mathcal{H}_{\nu\}} + \frac{3}{2}\mathcal{E}\Sigma_{\mu\nu} = \Re[\mathcal{F}_{\mu\nu}] \quad (59)$$

$$\left(\mathcal{L}_u + \frac{5}{2}\Sigma + \frac{1}{3}\theta\right)\mathcal{H}_{\bar{\mu}\bar{\nu}} - \epsilon_{(\mu}^\alpha\left(\mathcal{L}_n + 2\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{E}_{\nu)\alpha} + \epsilon_{\{\mu}^\alpha\delta_{|\alpha|}\mathcal{E}_{\nu\}} + \frac{3}{2}\mathcal{E}\epsilon_{(\mu}^\alpha\zeta_{\nu)\alpha} = \Im[\mathcal{F}_{\mu\nu}]. \quad (60)$$

The first-order energy-momentum source terms have been suitably defined in a complex form for later convenience as

$$\begin{aligned} \mathcal{F} := & -\frac{1}{2}(\mu + p)\Sigma - \frac{1}{3}\left(\mathcal{L}_n + 2\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{Q} + \frac{1}{6}\delta^\alpha\mathcal{Q}_\alpha - \frac{1}{2}\left(\mathcal{L}_u + \frac{1}{2}\Sigma + \frac{1}{3}\theta\right)\Pi \\ & + i\frac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha\Pi_\beta, \end{aligned} \quad (61)$$

$$\mathcal{G} := \frac{1}{3}\mathcal{L}_n\mu + \frac{1}{2}\mathcal{Q}\left(\Sigma - \frac{2}{3}\theta\right) - \frac{1}{2}\delta^\alpha\Pi_\alpha - \frac{1}{2}\left(\mathcal{L}_n + \frac{3}{2}\phi\right)\Pi - i\frac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha\mathcal{Q}_\beta, \quad (62)$$

$$\begin{aligned} \mathcal{F}_\mu := & -\frac{1}{2}\left[\mathcal{L}_u\Pi_{\bar{\mu}} + \left(\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{Q}_{\bar{\mu}} + \delta_\mu\mathcal{Q}\right] \\ & + i\frac{1}{2}\epsilon_\mu^\alpha\left[\frac{1}{3}\delta_\alpha(\mu + 3\Pi) - \left(\Sigma + \frac{1}{3}\theta\right)\mathcal{Q}_\alpha - \left(\mathcal{L}_n + \frac{1}{2}\phi\right)\Pi_\alpha\right], \end{aligned} \quad (63)$$

$$\begin{aligned} \mathcal{G}_\mu := & \frac{1}{3}\delta_\mu\left(\mu + \frac{3}{4}\Pi\right) - \frac{1}{4}\left(\Sigma + \frac{4}{3}\theta\right)\mathcal{Q}_\mu - \frac{1}{2}(\mathcal{L}_n + \phi)\Pi_{\bar{\mu}} - \frac{1}{2}\delta^\alpha\Pi_{\mu\alpha} \\ & + i\frac{1}{2}\epsilon_\mu^\alpha\left(\mathcal{L}_n\mathcal{Q}_\alpha - \delta_\alpha\mathcal{Q} + \frac{3}{2}\Sigma\Pi_\alpha\right), \end{aligned} \quad (64)$$

$$\begin{aligned} \mathcal{F}_{\mu\nu} := & -\frac{1}{2}\delta_{\{\mu}\mathcal{Q}_{\nu\}} - \frac{1}{2}\left(\mathcal{L}_u + \frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\Pi_{\bar{\mu}\bar{\nu}} \\ & + i\frac{1}{2}\left[\epsilon_{\{\mu}^\alpha\delta_{|\alpha|}\Pi_{\nu\}} - \epsilon_{(\mu}^\alpha\left(\mathcal{L}_n - \frac{1}{2}\phi\right)\Pi_{\bar{\nu})\alpha}\right]. \end{aligned} \quad (65)$$

The first-order GEM system (51)-(60) generalize those given in [1] in two significant ways; they generalize from the Schwarzschild perturbations towards an arbitrary vacuum LRS class II space-time and they also generalize from the vacuum energy-momentum perturbations towards a full energy-momentum perturbation. Furthermore, a very recent independent study of these equations for LRS space-times has been carried out in [15]. We have also taken a lot of care to ensure that all quantities are gauge-invariant; for example, the first-order term in (51),  $\delta[(\mathcal{L}_n + \frac{3}{2}\phi)\mathcal{E}]$ , is gauge-invariant as its corresponding background term vanishes according to (16), i.e.  $(\mathcal{L}_n + \frac{3}{2}\phi)\mathcal{E} = 0$ . However, we now choose to rewrite (51)-(54) in terms of the 2-gradient quantity  $X_\mu$

defined in (26). Thus, new complex variables are chosen according to the invariance properties of the 1+3 GEM system discussed above and, without loss of generality, we write the GEM system in a new 1+1+2 complex form,

$$\left(\mathcal{L}_n + \frac{3}{2}\phi\right)\mathcal{C}_\mu + \delta_\mu\delta^\alpha\Phi_\alpha + \frac{3}{2}\mathcal{E}\left[Y_\mu - \phi a_\mu - 2\left(\Sigma - \frac{2}{3}\theta\right)\epsilon_\mu{}^\alpha\Omega_\alpha + i2\delta_\mu\Omega\right] = \delta_\mu\mathcal{G}, \quad (66)$$

$$\left(\mathcal{L}_u - \frac{3}{2}\Sigma + \theta\right)\mathcal{C}_{\bar{\mu}} + i\delta_\mu(\epsilon^{\alpha\beta}\delta_\alpha\Phi_\beta) - \frac{3}{2}\mathcal{E}\left[\mathcal{A}_\mu\left(\Sigma - \frac{2}{3}\theta\right) + \phi(\Sigma_\mu - \epsilon_\mu{}^\alpha\Omega_\alpha + \alpha_\mu) + W_\mu - i2\delta_\mu\xi\right] = \delta_\mu\mathcal{F}, \quad (67)$$

$$(\mathcal{L}_n + \phi)\Phi_{\bar{\mu}} + \delta^\alpha\Phi_{\mu\alpha} - \frac{1}{2}\delta(\delta_\mu\Phi) - i\frac{3}{2}\Sigma\epsilon_\mu{}^\alpha\Phi_\alpha + \frac{3}{2}\mathcal{E}\Lambda_\mu = \mathcal{G}_\mu, \quad (68)$$

$$\left(\mathcal{L}_u - \Sigma + \frac{2}{3}\theta\right)\Phi_{\bar{\mu}} + i\epsilon_\mu{}^\alpha\delta^\beta\Phi_{\alpha\beta} + i\frac{1}{2}\epsilon_\mu{}^\alpha[\mathcal{C}_\alpha - (2\mathcal{A} - \phi)\Phi_\alpha] + \frac{3}{2}\mathcal{E}\Upsilon_\mu = \mathcal{F}_\mu, \quad (69)$$

$$\left(\mathcal{L}_u + \frac{5}{2}\Sigma + \frac{1}{3}\theta\right)\Phi_{\bar{\mu}\bar{\nu}} - i\epsilon_{(\mu}{}^\alpha\left(\mathcal{L}_n + 2\mathcal{A} - \frac{1}{2}\phi\right)\Phi_{\nu)\alpha} + i\epsilon_{\{\mu}{}^\alpha\delta_{|\alpha|}\Phi_{\nu\}} + \frac{3}{2}\mathcal{E}\Lambda_{\mu\nu} = \mathcal{F}_{\mu\nu}, \quad (70)$$

where

$$\mathcal{C}_\mu := X_\mu + i\delta_\mu\mathcal{H}, \quad \Phi_\mu := \mathcal{E}_\mu + i\mathcal{H}_\mu \quad \text{and} \quad \Phi_{\mu\nu} := \mathcal{E}_{\mu\nu} + i\mathcal{H}_{\mu\nu}. \quad (71)$$

Furthermore, whilst constructing these complex equations, several other terms naturally combine and therefore, 3 new complex definitions are

$$\Upsilon_\mu := \alpha_\mu + i\epsilon_\mu{}^\alpha\mathcal{A}_\alpha, \quad \Lambda_\mu := a_\mu + i\epsilon_\mu{}^\alpha(\Sigma_\alpha + \epsilon_\alpha{}^\beta\Omega_\beta) \quad \text{and} \quad \Lambda_{\mu\nu} := \Sigma_{\mu\nu} + i\epsilon_{(\mu}{}^\alpha\zeta_{\nu)\alpha}. \quad (72)$$

In Section 5 we will use the complex GEM system (66)-(70) to fully decouple the complex GEM 2-tensor,  $\Phi_{\mu\nu}$ , from all the remaining 1+1+2 quantities.

#### 4.3. The 1+1+2 Ricci Identities

The Ricci identities for both  $u^\mu$  and  $n^\mu$  are defined conveniently as

$$Q_{\mu\nu\sigma} := 2\nabla_{[\mu}\nabla_{\nu]}u_\sigma - R_{\mu\nu\sigma\tau}u^\tau = 0, \quad (73)$$

$$R_{\mu\nu\sigma} := 2\nabla_{[\mu}\nabla_{\nu]}n_\sigma - R_{\mu\nu\sigma\tau}n^\tau = 0, \quad (74)$$

where  $R_{\mu\nu\sigma\tau}$  is the Riemann tensor. We now linearize these, reduce them to 1+1+2 form and categorize them into constraint, propagation, transportation and evolution equations. We also make two new definitions for combinations that arise quite frequently,

$$\lambda_\mu := \Sigma_\mu - \epsilon_\mu{}^\alpha\Omega_\mu \quad \text{and} \quad v_\mu := \Sigma_\mu + \epsilon_\mu{}^\alpha\Omega_\mu, \quad (75)$$

such that the following system can be written in a more readable form.

- Constraint equations¶

$$W_\mu + \phi\lambda_\mu + 2\delta^\alpha\Sigma_{\mu\alpha} + 2\epsilon_\mu{}^\alpha\mathcal{H}_\alpha + 2\epsilon_\mu{}^\alpha\delta_\alpha\Omega = -\mathcal{Q}_\mu, \quad (76)$$

¶ It is also possible to choose the complex conjugates, i.e.  $\Phi_{\mu\nu}^*$ ,  $\Phi_\mu^*$  and  $\Phi^*$  and the corresponding governing equations are simplify found by taking the complex conjugate of the equations governing  $\Phi_{\mu\nu}$ ,  $\Phi_\mu$  and  $\Phi$ .

¶ (76) from a combination of  $n^\mu u^\sigma R_{\mu\bar{\nu}\sigma} = 0$ ,  $N^{\mu\sigma}Q_{\mu\bar{\nu}\sigma} = 0$  and  $n^\mu n^\sigma Q_{\mu\bar{\nu}\sigma} = 0$ ; (77) from  $N^{\nu\sigma}R_{\bar{\mu}\nu\sigma} = 0$  and (78) from  $\epsilon^{\mu\nu}u^\sigma R_{\mu\nu\sigma} = 0$ .

$$Y_\mu - 2\epsilon_\mu{}^\alpha \delta_\alpha \xi - 2\delta^\alpha \zeta_{\mu\alpha} + 2\mathcal{E}_\mu + \left(\Sigma - \frac{2}{3}\theta\right)\lambda_\mu = -\Pi_\mu, \quad (77)$$

$$\epsilon^{\alpha\beta} \delta_\alpha \lambda_\beta - (2\mathcal{A} - \phi)\Omega + 3\xi\Sigma - \mathcal{H} = 0. \quad (78)$$

- Propagation equations<sup>+</sup>

$$\delta\left\{\left(\mathcal{L}_n + \frac{1}{2}\phi\right)\phi + \left(\Sigma + \frac{1}{3}\theta\right)\left(\Sigma - \frac{2}{3}\theta\right) + \mathcal{E}\right\} - \delta^\alpha a_\alpha = -\frac{2}{3}\mu - \frac{1}{2}\Pi, \quad (79)$$

$$\delta\left\{\mathcal{L}_n\left(\Sigma - \frac{2}{3}\theta\right) + \frac{3}{2}\phi\Sigma\right\} + \delta^\alpha v_\alpha = -\mathcal{Q}, \quad (80)$$

$$(\mathcal{L}_n + \phi)\xi - \left(\Sigma + \frac{1}{3}\theta\right)\Omega - \frac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha a_\beta = 0, \quad (81)$$

$$(\mathcal{L}_n - \mathcal{A} + \phi)\Omega + \delta^\alpha \Omega_\alpha = 0, \quad (82)$$

$$\mathcal{L}_n \lambda_{\bar{\mu}} + \frac{1}{2}\phi v_\mu - 2\mathcal{A}\epsilon_\mu{}^\alpha \Omega_\alpha - \delta_\mu\left(\Sigma + \frac{1}{3}\theta\right) + \frac{3}{2}\Sigma a_\mu - \epsilon_\mu{}^\alpha \mathcal{H}_\alpha = -\frac{1}{2}\mathcal{Q}_\mu, \quad (83)$$

$$\left(\mathcal{L}_n - \frac{1}{2}\phi\right)\Sigma_{\bar{\mu}\bar{\nu}} - \frac{3}{2}\Sigma\zeta_{\mu\nu} - \epsilon_{(\mu}{}^\alpha \mathcal{H}_{\nu)\alpha} - \delta_{\{\mu}v_{\nu\}} = 0, \quad (84)$$

$$\mathcal{L}_n \zeta_{\bar{\mu}\bar{\nu}} - \left(\Sigma + \frac{1}{3}\theta\right)\Sigma_{\mu\nu} + \mathcal{E}_{\mu\nu} - \delta_{\{\mu}a_{\nu\}} = -\frac{1}{2}\Pi_{\mu\nu}. \quad (85)$$

- Transportation<sup>\*</sup>

$$\delta\left\{\left(\mathcal{L}_u + \Sigma + \frac{1}{3}\theta\right)\left(\Sigma + \frac{1}{3}\theta\right) - (\mathcal{L}_n + \mathcal{A})\mathcal{A} + \mathcal{E}\right\} = -\frac{1}{6}(\mu + 3p - 3\Pi), \quad (86)$$

$$\left(\mathcal{L}_u + \Sigma + \frac{1}{3}\theta\right)v_{\bar{\mu}} - \left(\mathcal{L}_n + \mathcal{A} - \frac{1}{2}\phi\right)\mathcal{A}_{\bar{\mu}} - \mathcal{A}a_\mu + \frac{3}{2}\Sigma\alpha_\mu + \mathcal{E}_\mu = \frac{1}{2}\Pi_\mu, \quad (87)$$

$$\left(\mathcal{L}_u + \frac{3}{2}\Sigma\right)a_{\bar{\mu}} - (\mathcal{L}_n + \mathcal{A})\alpha_{\bar{\mu}} - \left(\mathcal{A} - \frac{1}{2}\phi\right)v_\mu + \left(\Sigma + \frac{1}{3}\theta\right)\mathcal{A}_\mu - \epsilon_\mu{}^\alpha \mathcal{H}_\alpha = -\frac{1}{2}\mathcal{Q}_\mu. \quad (88)$$

- Evolution equations<sup>‡</sup>

$$\delta\left\{\left(\mathcal{L}_u - \frac{1}{2}\Sigma + \frac{1}{3}\theta\right)\phi + \mathcal{A}\left(\Sigma - \frac{2}{3}\theta\right)\right\} - \delta^\gamma \alpha_\gamma = \mathcal{Q}, \quad (89)$$

$$\delta\left\{\left(\mathcal{L}_u - \frac{1}{2}\Sigma + \frac{1}{3}\theta\right)\left(\Sigma - \frac{2}{3}\theta\right) + \mathcal{A}\phi + \mathcal{E}\right\} + \delta^\alpha \mathcal{A}_\alpha = \frac{1}{3}\left(\mu + 3p + \frac{3}{2}\Pi\right), \quad (90)$$

$$\left(\mathcal{L}_u - \frac{1}{2}\Sigma + \frac{1}{3}\theta\right)\xi - \frac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha \alpha_\beta - \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega - \frac{1}{2}\mathcal{H} = 0, \quad (91)$$

$$\left(\mathcal{L}_u - \Sigma - \frac{2}{3}\theta\right)\Omega - \mathcal{A}\xi - \frac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha \mathcal{A}_\beta = 0, \quad (92)$$

$$(\mathcal{L}_u + \theta)\lambda_{\bar{\mu}} - Z_\mu - \left(\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{A}_\mu + \frac{3}{2}\Sigma\alpha_\mu + \mathcal{E}_\mu = \frac{1}{2}\Pi_\mu, \quad (93)$$

$$\left(\mathcal{L}_u + \frac{1}{2}\Sigma - \frac{1}{3}\theta\right)\zeta_{\bar{\mu}\bar{\nu}} - \left(\mathcal{A} - \frac{1}{2}\phi\right)\Sigma_{\mu\nu} - \epsilon_{(\mu}{}^\alpha \mathcal{H}_{\nu)\alpha} - \delta_{\{\mu}\alpha_{\nu\}} = 0, \quad (94)$$

$$\mathcal{L}_u \Sigma_{\bar{\mu}\bar{\nu}} - \mathcal{A}\zeta_{\mu\nu} - \delta_{\{\mu}\mathcal{A}_{\nu\}} + \mathcal{E}_{\mu\nu} = \frac{1}{2}\Pi_{\mu\nu}. \quad (95)$$

<sup>+</sup> (79) from  $n^\mu N^{\nu\sigma} R_{\mu\nu\sigma} = 0$ ; (80) from  $n^\mu N^{\nu\sigma} Q_{\mu\nu\sigma} = 0$ ; (81) from  $n^\mu \epsilon^{\nu\sigma} R_{\mu\nu\sigma} = 0$ ; (82) from  $\epsilon^{\mu\nu\sigma} Q_{\mu\nu\sigma} = 0$ ; (83) from  $D^\alpha \sigma_{\mu\alpha}$  equation and  $n^\mu u^\sigma R_{\mu\bar{\nu}\sigma} = 0$ ; (84) from  $n^\mu Q_{\mu(\bar{\nu}\bar{\sigma})} = 0$ ; (85) from  $n^\mu R_{\mu(\bar{\nu}\bar{\sigma})} = 0$ .

<sup>\*</sup> (86) from  $u^\mu n^\nu u^\sigma R_{\mu\nu\sigma}$ ; (87) from  $n^\mu u^\nu N_\sigma{}^\gamma Q_{\mu\nu\gamma} = 0$ ; (88) from  $u^\alpha n^\beta R_{\alpha\beta\bar{\mu}} = 0$ .

<sup>‡</sup> (89) from  $u^\mu N^{\nu\sigma} R_{\mu\nu\sigma} = 0$ ; (90) from  $u^\mu N^{\nu\sigma} Q_{\mu\nu\sigma} = 0$ ; (91) from  $u^\mu \epsilon^{\nu\sigma} R_{\mu\nu\sigma} = 0$ ; (92) from  $u^\mu \epsilon^{\nu\sigma} Q_{\mu\nu\sigma} = 0$ ; (93) from  $u^\mu n^\sigma N_\nu{}^\alpha Q_{\mu\alpha\sigma} = 0$ ; (94) from  $u^\mu R_{\mu(\bar{\nu}\bar{\sigma})} = 0$ ; (95) from  $u^\mu Q_{\mu(\bar{\nu}\bar{\sigma})} = 0$ .

Similarly, these 1+1+2 Ricci identities (76)-(95) are again a significant generalization of the results in [1]. They now include full energy-momentum sources and moreover, they are for arbitrary vacuum LRS class II space-times. Moreover, the very recent independent study by Clarkson [15] presents the equations for LRS space-times. For the subsequent decoupling of the complex GEM 2-tensor, we require evolution, transportation and propagation equations for the complex variables defined in (72)<sup>††</sup>

$$\begin{aligned} & \left( \mathcal{L}_u + \frac{3}{2} \Sigma \right) \Lambda_{\bar{\mu}} - (\mathcal{L}_n + \mathcal{A}) \Upsilon_{\bar{\mu}} + i \epsilon_{\mu}^{\alpha} \Phi_{\alpha} - i \mathcal{A} \epsilon_{\mu}^{\alpha} \Lambda_{\alpha} + \frac{1}{2} \phi (v_{\mu} + i \epsilon_{\mu}^{\alpha} \mathcal{A}_{\alpha}) \\ & + \left( \Sigma + \frac{1}{3} \theta \right) \mathcal{A}_{\mu} - i \frac{1}{2} \left( \Sigma - \frac{2}{3} \theta \right) \epsilon_{\mu}^{\alpha} v_{\alpha} + i \frac{3}{2} \Sigma \epsilon_{\mu}^{\alpha} \alpha_{\alpha} = -\frac{1}{2} (\mathcal{Q}_{\mu} - i \epsilon_{\mu}^{\alpha} \Pi_{\alpha}), \end{aligned} \quad (96)$$

$$\begin{aligned} & \mathcal{L}_u \Lambda_{\bar{\mu}\bar{\nu}} + \Phi_{\mu\nu} - i \mathcal{A} \epsilon_{(\mu}^{\alpha} \Lambda_{\nu)\alpha} + i \frac{1}{2} \phi \epsilon_{(\mu}^{\alpha} \Sigma_{\nu)\alpha} \\ & + i \frac{1}{2} \left( \Sigma - \frac{2}{3} \theta \right) \epsilon_{(\mu}^{\alpha} \zeta_{\nu)\alpha} - i \epsilon_{\{\mu}^{\alpha} \delta_{\nu\}} \Upsilon_{\alpha} = \frac{1}{2} \Pi_{\mu\nu}, \end{aligned} \quad (97)$$

$$\begin{aligned} & \mathcal{L}_n \Lambda_{\bar{\mu}\bar{\nu}} + i \epsilon_{(\mu}^{\alpha} \Phi_{\nu)\alpha} - i \left( \Sigma + \frac{1}{3} \theta \right) \epsilon_{(\mu}^{\alpha} \Sigma_{\nu)\alpha} - \frac{3}{2} \Sigma \zeta_{\mu\nu} \\ & - \frac{1}{2} \phi \Sigma_{\mu\nu} - i \epsilon_{\{\mu}^{\alpha} \delta_{\nu\}} \Lambda_{\alpha} = -i \frac{1}{2} \epsilon_{(\mu}^{\alpha} \Pi_{\nu)\alpha}. \end{aligned} \quad (98)$$

#### 4.4. Commutation relationships

Finally, we present how the various derivatives defined in this paper commute and generalize the results from [2],

$$\left( \mathcal{L}_u + \Sigma + \frac{1}{3} \theta \right) \mathcal{L}_n \Phi_{\bar{\mu} \dots \bar{\nu}} - (\mathcal{L}_n + \mathcal{A}) \mathcal{L}_u \Phi_{\bar{\mu} \dots \bar{\nu}} = 0, \quad (99)$$

$$\mathcal{L}_u \delta_{\sigma} \Phi_{\bar{\mu} \dots \bar{\nu}} - \delta_{\sigma} \mathcal{L}_u \Phi_{\bar{\mu} \dots \bar{\nu}} = 0, \quad (100)$$

$$\mathcal{L}_n \delta_{\sigma} \Phi_{\bar{\mu} \dots \bar{\nu}} - \delta_{\sigma} \mathcal{L}_n \Phi_{\bar{\mu} \dots \bar{\nu}} = 0. \quad (101)$$

where  $\Phi_{\mu \dots \nu}$  represents a first-order scalar, first-order 2-vector and a first-order 2-tensor. The commutators not only play a vital role in decoupling the equations at hand, they also provide a rigorous test that the equations present here are correct and accurate. Every equation (61)-(70) and (76)-(98) has been checked to satisfy all of the commutator relationships (99)-(100) and this is inclusive of careful checks of all energy-momentum source terms (61)-(65).

### 5. Decoupling the complex GEM 2-tensor and its tensor harmonic amplitudes

We use the complex 1+1+2 Bianchi identities (66)-(70) to construct a new, covariant and gauge-invariant, equation governing the first-order complex GEM 2-tensor  $\Phi_{\mu\nu}$ . This is with a complete description of the, covariant and gauge-invariant, first-order energy-momentum sources. It begins by taking the Lie derivative with respect to  $u^{\mu}$

<sup>††</sup>(96) from (87) and (88); (97) from (94), (95); (98) from (84) and (85)

of (70). It is then required to use the commutation relationships (99)-(100) followed by substitutions of (68) through to (70). Finally, (97) and (98) are used for further simplifications to obtain

$$[(\mathcal{L}_u + \theta)\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + \phi)\mathcal{L}_n - V]\Phi_{\mu\nu} - i\epsilon_{(\mu}{}^\alpha [(4\mathcal{A} - 2\phi)\mathcal{L}_u - 6\Sigma\mathcal{L}_n + U]\Phi_{\nu)\alpha} = \mathcal{M}_{\mu\nu}. \quad (102)$$

The two background scalars related to the potential, and the first-order energy-momentum source, have been defined respectively

$$V := \delta^2 + 8\mathcal{E} - 4\mathcal{A}^2 + 4\mathcal{A}\phi - \phi^2 + 9\Sigma^2 - 3\Lambda, \quad (103)$$

$$U := 2\left(\mathcal{L}_u - \Sigma + \frac{2}{3}\theta\right)\mathcal{A} - 3\left(\mathcal{L}_n + \frac{7}{6}\phi\right)\Sigma - \frac{2}{3}\theta\phi - 2\Lambda, \quad (104)$$

$$\mathcal{M}_{\mu\nu} := \left(\mathcal{L}_u - \frac{5}{2}\Sigma + \frac{2}{3}\theta\right)\mathcal{F}_{\bar{\mu}\bar{\nu}} + i\epsilon_{(\mu}{}^\alpha \left(\mathcal{L}_n - \mathcal{A} + \frac{3}{2}\phi\right)\mathcal{F}_{\nu)\alpha} - i\epsilon_{\{\mu}{}^\alpha \delta_{|\alpha|} \mathcal{F}_{\nu\}} - \delta_{\{\mu} \mathcal{G}_{\nu\}}. \quad (105)$$

It was possible to eliminate all Lie derivatives in  $V$  and write it explicitly as algebraic combinations of the background LRS class II scalars. However, the Lie derivatives in the other potential term,  $U$ , cannot be reduced any further because there is no evolution equation for  $\mathcal{A}$ .

Thus (102) demonstrates that, for arbitrary vacuum LRS class II space-times, the complex GEM 2-tensor decouples from the remaining GEM and 1+1+2 quantities. We next show how this 2-tensor decouples further by using a tensor harmonic expansion, but we first take a closer inspection of the energy-momentum source,  $\mathcal{M}_{\mu\nu}$ ,

$$\begin{aligned} \mathcal{M}_{\mu\nu} = & -\frac{1}{2}\left\{\left(\mathcal{L}_u - 2\Sigma + \frac{1}{3}\theta\right)\mathcal{L}_u\Pi_{\mu\nu} + (\mathcal{L}_n - \mathcal{A} + \phi)\mathcal{L}_n\Pi_{\mu\nu} - \mathcal{M}\Pi_{\mu\nu} - 2\delta_{\{\mu}\delta^\alpha\Pi_{\nu\}\alpha}\right\} \\ & + 2(\mathcal{L}_n + \phi)\delta_{\{\mu}\Pi_{\nu\}} + 2\left(\Sigma + \frac{1}{3}\theta\right)\delta_{\{\mu}\mathcal{Q}_{\nu\}} + \frac{1}{2}\delta_{\{\mu}\delta_{\nu\}}(p + 2\Pi), \\ & + i\epsilon_{\{\mu}{}^\alpha\left\{-\left(\mathcal{L}_u - \frac{1}{2}\Sigma + \frac{1}{3}\theta\right)\mathcal{L}_n\Pi_{\nu)\alpha} + \left(\mathcal{A} - \frac{1}{2}\phi\right)\mathcal{L}_u\Pi_{\nu)\alpha} + \phi\left(\Sigma - \frac{2}{3}\theta\right)\Pi_{\nu)\alpha}\right. \\ & \left.+ \left(\mathcal{L}_u - 2\Sigma + \frac{1}{3}\theta\right)\delta_{\nu)\alpha} - (\mathcal{L}_n - \mathcal{A} + \phi)\delta_{\nu)\alpha}\mathcal{Q}_\alpha + \delta_{\nu)\alpha}\mathcal{Q}_\alpha\right\}, \end{aligned} \quad (106)$$

where

$$\mathcal{M} = \frac{1}{2}\left(\Sigma - \frac{2}{3}\theta\right)^2 + \frac{1}{2}\mathcal{A}\phi + \frac{1}{2}\phi^2 - \frac{1}{2}\mathcal{E}. \quad (107)$$

It is interesting to see which energy-momentum terms play an important role in the evolution and propagation of the complex GEM 2-tensor. By considering the “principle part”, or the parts which involve second-order Lie derivatives, it seems that the first-order anisotropic stress may have a predominate influence here.

### 5.1. Decoupling the complex GEM 2-tensor harmonic amplitudes

The complex GEM tensor,  $\Phi_{\mu\nu}$ , and the energy-momentum source,  $\mathcal{M}_{\mu\nu}$ , are expanded using tensor harmonics according to

$$\Phi_{\mu\nu} = \Phi_{\text{T}} Q_{\mu\nu} + \bar{\Phi}_{\text{T}} \bar{Q}_{\mu\nu} \quad \text{and} \quad \mathcal{M}_{\mu\nu} = \mathcal{M}_{\text{T}} Q_{\mu\nu} + \bar{\mathcal{M}}_{\text{T}} \bar{Q}_{\mu\nu}.$$

Consequently, (102) results in two coupled equations of the form

$$\begin{aligned} & \left[ \left( \mathcal{L}_u - 2\Sigma + \frac{7}{3}\theta \right) \mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi) \mathcal{L}_n - \tilde{V} \right] \Phi_T \\ & + i \left[ 6\Sigma \mathcal{L}_n - (4\mathcal{A} - 2\phi) \mathcal{L}_u - \tilde{U} \right] \bar{\Phi}_T = \mathcal{M}_T, \end{aligned} \quad (108)$$

$$\begin{aligned} & \left[ \left( \mathcal{L}_u - 2\Sigma + \frac{7}{3}\theta \right) \mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi) \mathcal{L}_n - \tilde{V} \right] \bar{\Phi}_T \\ & - i \left[ 6\Sigma \mathcal{L}_n - (4\mathcal{A} - 2\phi) \mathcal{L}_u - \tilde{U} \right] \Phi_T = \bar{\mathcal{M}}_T, \end{aligned} \quad (109)$$

where new potential terms are defined

$$\begin{aligned} \tilde{V} &:= -\frac{k^2}{r^2} + 2\mathcal{E} - 4\mathcal{A}^2 + 4\mathcal{A}\phi + \frac{3}{2}\phi^2 + \frac{13}{2}\Sigma^2 - \frac{10}{9}\theta^2 + \frac{10}{3}\Sigma\theta, \\ \tilde{U} &:= 2\left(\mathcal{L}_u - 3\Sigma + 2\theta\right)\mathcal{A} - 3\left(\mathcal{L}_n + \frac{5}{2}\phi\right)\Sigma - 2\theta\phi. \end{aligned} \quad (110)$$

By inspecting the coupled system (108) and (109), it is clear that they are invariant under the simultaneous transformation of  $\Phi_T \rightarrow \bar{\Phi}_T$  and  $\bar{\Phi}_T \rightarrow -\Phi_T$ , and similarly for the sources,  $\mathcal{M}_T \rightarrow \bar{\mathcal{M}}_T$  and  $\bar{\mathcal{M}}_T \rightarrow -\mathcal{M}_T$ . Thus, the coupled system (108)-(109) is precisely of the form as discussed at the beginning of Section 4.2. Therefore, they will decouple quite naturally by constructing two new complex dependent variables,

$$\Phi_+ := \Phi_T + i\bar{\Phi}_T \quad \text{and} \quad \Phi_- := \Phi_T - i\bar{\Phi}_T. \quad (111)$$

We also define a new complex energy-momentum source  $\mathcal{M}_\pm := \mathcal{M}_T \pm i\bar{\mathcal{M}}_T$  and potential  $V_\pm := \tilde{V} \pm \tilde{U}$ , where the “ $\pm$ ” is relative. Therefore, by taking complex combinations of (108) and (109), we find two new decoupled equations given by

$$\left\{ \left[ \mathcal{L}_u - 2\Sigma + \frac{7}{3}\theta + (2\phi - 4\mathcal{A}) \right] \mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi - 6\Sigma) \mathcal{L}_n - V_+ \right\} \Phi_+ = \mathcal{M}_+, \quad (112)$$

$$\left\{ \left[ \mathcal{L}_u - 2\Sigma + \frac{7}{3}\theta - (2\phi - 4\mathcal{A}) \right] \mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi + 6\Sigma) \mathcal{L}_n - V_- \right\} \Phi_- = \mathcal{M}_-. \quad (113)$$

It is vital to point out here that, since the covariant differential operators in (112)-(113) are purely real, by taking the real and imaginary parts separately there are actually four *real* decoupled quantities. It is now of interest to see how  $\Phi_\pm$  relates back to the real GEM 2-tensor harmonic amplitudes. The GEM 2-tensors are expanded according to

$$\mathcal{E}_{\mu\nu} = \mathcal{E}_T Q_{\mu\nu} + \bar{\mathcal{E}}_T \bar{Q}_{\mu\nu} \quad \text{and} \quad \mathcal{H}_{\mu\nu} = \mathcal{H}_T Q_{\mu\nu} + \bar{\mathcal{H}}_T \bar{Q}_{\mu\nu}. \quad (114)$$

Here, the polar perturbations are  $\mathcal{E}_T$  and  $\bar{\mathcal{H}}_T$  whereas the axial perturbations are  $\bar{\mathcal{E}}_T$  and  $\mathcal{H}_T$ . Moreover, a full categorization of all the harmonic amplitudes of the 1+1+2 dependent variables into polar and axial perturbations is presented in [1]. The definition (71) now implies,

$$\Phi_T := \mathcal{E}_T + i\mathcal{H}_T \quad \text{and} \quad \bar{\Phi}_T := \bar{\mathcal{E}}_T + i\bar{\mathcal{H}}_T, \quad (115)$$

and by subsequently using (111) we find,

$$\Phi_+ = (\mathcal{E}_T - \bar{\mathcal{H}}_T) + i(\bar{\mathcal{E}}_T + \mathcal{H}_T) \quad \text{and} \quad \Phi_- = (\mathcal{E}_T + \bar{\mathcal{H}}_T) - i(\bar{\mathcal{E}}_T - \mathcal{H}_T). \quad (116)$$

Thus, the four precise combinations of the four real GEM 2-tensor harmonic amplitudes which decouple are,

$$\text{Decoupled polar perturbations: } \{\mathcal{E}_T + \bar{\mathcal{H}}_T, \mathcal{E}_T - \bar{\mathcal{H}}_T\}, \quad (117)$$

$$\text{Decoupled axial perturbations: } \{\mathcal{H}_T + \bar{\mathcal{E}}_T, \mathcal{H}_T - \bar{\mathcal{E}}_T\}. \quad (118)$$

Moreover, it is clear that if the 4 decoupled quantities are known, then simple linear combinations will reveal each of  $\mathcal{E}_T$ ,  $\bar{\mathcal{H}}_T$ ,  $\mathcal{H}_T$  and  $\bar{\mathcal{E}}_T$ .

## 6. Summary

This paper discussed covariant and gauge-invariant gravitational and energy-momentum perturbations on arbitrary vacuum LRS class II space-times. We showed how particular combinations of the first-order GEM quantities decouple at two different levels. The first was a complex tensorial equation governing the complex GEM 2-tensor  $\Phi_{\mu\nu}$  (102). The second involved a tensor harmonic expansion of the GEM 2-tensors and resulted in four *real* equations (112)-(113). Of particular interest is that we have found the precise combinations of the GEM 2-tensor harmonic amplitudes that decouple, and these were separated out into polar and axial perturbations according to (117)-(118).

## References

- [1] Clarkson C and Barrett R 2003 *Class. Quantum Grav.* **20** 3855-84
- [2] Betschart G and Clarkson C 2004 *Class. Quantum Grav.* **21** 5587-607
- [3] Ellis G F R 1967 *J. Math. Phys.* **8** 1171
- [4] Stewart J M and Ellis G F R 1968 *J. Math. Phys.* **9** 1072
- [5] Elst H and Ellis G F R 1996 *Class. Quantum Grav.* **13** 1099-127
- [6] Regge T and Wheeler J 1957 *Phys. Rev.* **108** 1063
- [7] Burston R B and Lun A W C 2007 *1+1+2 electromagnetic perturbations on LRS space-times: Regge-Wheeler and Bardeen-Press equations* submitted to *Class. Quantum Grav.*
- [8] Burston R B 2007 *1+1+2 electromagnetic perturbations on LRS class II space-times: Decoupling vector and scalar harmonic amplitudes* submitted to *Class. Quantum Grav.*
- [9] Bel L 1958 *C. R. Acad. Sci.* **247** 1094
- [10] Maartens R and Bassett B 1998 *Class. Quantum Grav.* **15** 705-17
- [11] Sachs R 1964 *Relativity, groups and topology* (eds DeWitt B and DeWitt C)
- [12] Stewart J M and Walker M 1974 *Proc. R. Soc.* **341** 49-74
- [13] Ehlers J 1993 *Gen. Rel. Grav.* **25** 1225-66
- [14] D'Inverno R (1998) *Introducing Einstein's Relativity* 69-72
- [15] Clarkson C 2007 *arXiv:0708.139v1 [gr-qc]*